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# Disordered Ground States for Classical Discrete-State Problems in One Dimension

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It is known that one-dimensional lattice problems with a discrete, finite set of states per site "generically" have periodic ground states (GSs). We consider slightly less generic cases, in which the Hamiltonian is constrained by either spin ($S$) or spatial ($I$) inversion symmetry (or both). We show that such constraints give rise to the possibility of *disordered* GSs over a finite fraction of the coupling-parameter space—that is, without invoking any nongeneric "fine tuning" of coupling constants, beyond that arising from symmetry. We find that such disordered GSs can arise for many values of the number of states $k$ at each site and the range $r$ of the interaction. The Ising ($k=2$) case is the least prone to disorder: $I$ symmetry allows for disordered GSs (without fine tuning) only for $r \geqslant 5$, while $S$ symmetry "never" gives rise to disordered GSs.

**KEY WORDS:** Ising models; disorder; ground states; directed graphs; polytypes; third law.

## 1. INTRODUCTION

The problem of order vs. disorder permeates all of condensed-matter and statistical physics. If we ignore thermal fluctuations by setting $T=0$, and quantum fluctuations as well by viewing matter as composed of massive units interacting via effective classical potentials, we have a simpler problem which is still nontrivial. Here we want to consider the simplest subproblem of this class: we restrict our units to lie on a one-dimensional chain, and allow them only a finite, discrete set of states, whose number we call $k$. We take the (integer) range of the interaction among the units to be

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$r$, but do not restrict the interactions to two-body terms. (We will call the units "spins.") The Hamiltonian is then of the form

$$H = \sum_i f(\sigma_i, \sigma_{i+1}, ..., \sigma_{i+r}) \tag{1}$$

where $\sigma_i$ (the spin at site $i$) has $k$ states which we label 0, 1,..., $(k-1)$, and we assume an infinite chain.

The question of ordered/disordered ground states (GSs) for this problem has already been answered in principle. Radin and Schulman (RS)[1] showed that (i) a nondegenerate GS is periodic and (ii) in the case of degenerate GSs, there always exists at least one periodic GS. (See also Teubner[2] for a different presentation of the same results for the Ising case $k=2$.) In case (i) the period of the GS is $\leqslant k^r$; in case (ii), one can specify a shortest period periodic GS whose period is again $\leqslant k^r$.

A simple and pictorial understanding of these results is possible by embodying the information contained in the Hamiltonian $H$ in a directed graph $G_r^{(k)}$ (where $k$ and $r$ have the same meanings as above). This is done as follows:[2,3] nodes of the graph are sets of $r$ spins, each taking one of the $k$ values. A directed arc points from node $\mathcal{N}_1$ to node $\mathcal{N}_2$ whenever the rightmost $(r-1)$ values of $\mathcal{N}_1$ agree with the leftmost $(r-1)$ values of $\mathcal{N}_2$. The arc itself may then be uniquely labeled with $(r+1)$ sequential spin values, which allows us to associate a unique weight (energy) to the arc. Specifically, we can take the weight of the arc joining the node $(\sigma_i, \sigma_{i+1}, ..., \sigma_{i+r-1})$ to the node $(\sigma_{i+1}, \sigma_{i+2}, ..., \sigma_{i+r})$ to be $f(\sigma_i, \sigma_{i+1}, ..., \sigma_{i+r})$. The graph $G_r^{(k)}$ may be viewed as a "machine" which reads a string of spins as input, and outputs a string of arc weights, whose sum is the energy of the string; hence, in this sense, the graph $G_r^{(k)}$ represents the Hamiltonian $H_r^{(k)}$.

The graph $G_r^{(k)}$ has $k^r$ nodes and $k^{r+1}$ arcs. Any configuration of an infinite chain of spins thus must involve repeated cycles of arcs in $G_r^{(k)}$. All cycles in $G_r^{(k)}$ may be decomposed into "simple cycles" (SCs)[4] having the property of non-self-intersection. The result of Radin and Schulman then appears in this approach in the following form: (i) If there is a unique SC of $G_r^{(k)}$ with the lowest weight per spin, then a repetition of that SC gives the nondegenerate GS, whose period (the length of the SC) is $\leqslant$ the number of nodes of the graph, i.e., $k^r$. (ii) If there are two or more SCs with the lowest weight per spin, then there is always a GS consisting of repetitions of only one of them, whose period is again $\leqslant k^r$.

This logic does not allow a reduction of the upper bound of RS. The graphs $G_r^{(k)}$ are known as de Bruijn graphs;[5,6] and it is known that the set of SCs always includes a "Hamiltonian cycle," that is, one visiting all the nodes. Furthermore, it is clear that case (ii) is "rare": in general—that is, without fine tuning of the couplings to precise values—the Hamiltonian $H$ does not give degenerate SCs.

We also note that, in the exceptional case of degenerate (and minimal-weight) SCs, the degeneracy may give rise to *disordered* GSs if the degenerate SCs share one or more nodes—since the different SCs may then be traversed in any arbitrary sequence with no energy cost.

Put briefly, we see that, barring fine tuning of the coupling parameters of $H$, one always has a periodic GS for a $k$-state model. Given such fine tuning, however, one may find an uncountable set of degenerate GS configurations; in this case, essentially all of these degenerate configurations are aperiodic.

Given this background, we note the following: sometimes, "fine tuning" is "generic." By this we mean simply that *symmetry* in fact does tune some parameters to precise values. This suggests the following question: might the symmetries of $H$ give rise to disordered, degenerate GSs—*without* any further fine tuning of parameters?

We offer an answer to this question here. The symmetries we consider are two: spin inversion ($S$) symmetry, and spatial inversion ($I$) symmetry. We choose these two because they are both (particularly the latter) very common in physical applications. (We should also point out that our entire analysis, like that of RS, assumes translational invariance of the Hamiltonian as well; put differently, we are seeking disordered GSs in the absence of quenched disorder in the Hamiltonian.)

## 2. GROUND STATES, DEGENERACY, AND DISORDER

We wish to study ground states of infinite chains of spins. Since the total energy of an infinite chain is not well defined, some conventions are needed, both for describing the configurations of an infinite chain and for comparing the energies of distinct configurations. A standard approach[1,7,8] is to compare configurations differing only in a finite number of spins; a ground-state configuration is then one whose energy cannot be lowered by changing any finite number of spins. Here we follow Teubner[2] in using an alternative approach: we wish to compare the energy densities of different configurations (confining our attention to those configurations for which the density, defined as the limit of the energy per spin as the chain length $N \to \infty$, is well defined). We then define a ground-state configuration as one for which the energy density is minimal.

It is well known[7,8] that the "standard" definition allows for multiple ground states; for example, "standard" ground states for the near-neighbor Ising ferromagnet include all configurations with a single domain wall, as well as the defectless ( + ) and ( − ) periodic configurations. The entropy density of these defected configurations is of course zero in the thermodynamic limit. If we instead use minimal energy density as our criterion for ground states, we again find multiple ground states for any reasonable Hamiltonian. In the Ising ferromagnet, for example, one must include any finite number of defective spins (which cannot affect the energy density), or even an infinite number, as long as the defective spins have a vanishing density in the limit $N \to \infty$. However, a simple argument[4] shows that the entropy density of defects of vanishing number density also vanishes. Thus the most important properties (energy and entropy densities, correlations) of the multiple ground states are equivalent to, and represented by, those of the ground states without defects. Hence in this paper (for example in the Introduction above), we sometimes, for brevity, refer to "the" ground state (GS) or ground states (GSs) for a given, specific Hamiltonian, by which we mean those which are free of defects. (Here, we may loosely define a "defect" as a string of spins which, if present with finite density, raises the energy density; defects will be defined more precisely below.)

We are interested in disordered ground states. Clearly, by "disorder" we do not mean the proliferation of defects (of vanishing density) which is implicit in our definition of ground states (as energy-density minimizers), and which occurs for even the simplest Hamiltonians. Instead, we impose two criteria for a disordered ground state. The first is that the entropy per spin of the ground-state configurations should be nonzero. The second criterion is that the first criterion should hold over a finite neighborhood of Hamiltonians; that is, we exclude disorder arising from the fine tuning of the coupling constants of the Hamiltonian to some precise values.

We add this second criterion because the coupling constants of a Hamiltonian are not generally under experimental control, and it is unreasonable to ask nature to give us precise values. The exception is of course the precision coming from symmetry. The result of Radin and Schulman tells us that we should "never" expect to see disordered ground states for 1D, $k$-state problems. However, if a symmetry is exact, then we

[4] Assume that the number of defects grows with increasing number of spins $N$ as some function $f(N)$, with the property $\lim_{N \to \infty}(f/N) = 0$. Viewing a spin configuration as a path through the graph $G_r^{(k)}$, we define a defect as a "bad" arc: that is, an arc not on that SC which lies at the vertex of $P_r^{(k)}$ minimizing $\mathscr{H}$. Then, each time a spin (arc) is added to the chain, there are either $k-1$ or $k$ bad choices. Hence the number of distinct lists of $f$ bad arcs is less than $k^f$. Assuming no freedom in choosing "good" arcs, there are $\binom{N}{(N-f)} = \binom{N}{f}$ choices for placement of each list of $f$ defects, so that the total degeneracy of $f$ defects is less than $W = k^f\binom{N}{f}$. The entropy density $s = (\ln W)/N$ then vanishes as $N \to \infty$. If we assume two degenerate SCs such that there are two choices of good arc sequences for each (long) sequence between defects, $W$ is multiplied by a prefactor which is less than $2^f$, and our conclusion $s \to 0$ is unchanged.

can expect exact relations between coupling constants, giving rise to exact degeneracy among periodic configurations, and thus to possible disordered ground states by our criteria. Hence, for the purposes of this paper, we modify our second criterion: we wish to confine the "finite neighborhood" of Hamiltonians to the space of symmetric Hamiltonians. That is, we will assume the symmetry is exact, and so holds for any variation of the couplings. With this modification, we apply our second criterion to *any* ground-state configuration, disordered or not: it must minimize the energy density over a finite neighborhood of symmetric Hamiltonians.

In seeking to minimize the Hamiltonian density of infinite chains, we will heavily use two tools. The first, the graph $G_r^{(k)}$, has been described above. We will find it useful to define and study a modified graph ${}^XG_r^{(k)}$ for the case that the Hamiltonian is invariant under the symmetry $X$ ($=S$, $I$, or a combination). Our construction of ${}^XG_r^{(k)}$ is chosen such that the simple cycles of this graph correspond to the set of defect-free ground-state configurations which may be realized for *any* possible Hamiltonian ${}^XH_r^{(k)}$ of the given $r$ and $k$ and symmetry $X$. This correspondence is identical to that shown by Teubner[2] for the case $k=2$ and $X=0$ (no symmetry).

Our second tool is the polytope $P_r^{(k)}$ and its projection ${}^XP_r^{(k)}$ onto an $X$-invariant subspace. This polytope arises as follows. We can write the Hamiltonian density as $\mathscr{H} = -\sum J_\alpha s_\alpha = -\mathbf{J}\cdot\mathbf{s}$. Here $s_\alpha$ is one of the set of spin correlations[2] of the form $\langle \sigma_i^{p_i}\sigma_{i+1}^{p_{i+1}}\cdots\sigma_{i+r}^{p_{i+r}}\rangle$, where $\alpha$ denotes the set $\{p_i\}$ [each of which, for $i>1$, may take the values $(0, 1,..., k-1)$; for $i=1$ we omit $p_1=0$ to avoid redundancy]. The angle brackets signify the average over the chain, taken in the limit $N\to\infty$. The coupling vector $\mathbf{J}$ is in general unconstrained; however, the correlation vector $\mathbf{s}$ is not. The correlations $s_\alpha$ may be written[2] as linear combinations of the *arc densities* $n_{\sigma_i, \sigma_{i+1},..., \sigma_{i+r}}$. The arc density $n_\sigma$ is the average occurrence of the arc $\sigma\equiv\{\sigma_1\sigma_2\cdots\sigma_{r+1}\}$ in a given configuration of spins. Clearly there is one of these for each of the $k^{r+1}$ arcs; however,[2] only $d=k^r(k-1)$ are independent. This latter number is thus the number of independent correlations $s_\alpha$; assuming $\mathscr{H}$ is written without redundant correlations, we find that the vectors $\mathbf{J}$ and $\mathbf{s}$ reside in a $d$-dimensional space. The correlations $n_\sigma$ obey the inequalities $0\leqslant n_\sigma\leqslant 1$. These inequalities give rise to inequalities for the correlations $s_\alpha$; in particular, the bound $n=0$ defines a hyperplane in $\mathbf{s}$ space, and the set of all such hyperplanes defines a convex polytope[2] $P_r^{(k)}$.

The simple cycles of the graph $G_r^{(k)}$ may be placed in one-to-one correspondence[2] with the vertices of the polytope $P_r^{(k)}$: an SC represents a point $x_{\mathrm{SC}}$ in $\mathbf{s}$ space, satisfying a number of equalities $n_\sigma=0$, such that any point in a neighborhood of $x_{\mathrm{SC}}$ violates one or more of those equalities (by including a finite density of arcs not in the SC). Hence $x_{\mathrm{SC}}$ represents the intersection of hyperplanes, such that fewer hyperplanes intersect in any allowed neighborhood of $x_{SC}$: it is a vertex of the convex polytope $P_r^{(k)}$. As such, it represents an extremum of $\mathscr{H} = -\mathbf{J} \cdot \mathbf{s}$ over a finite neighborhood of $\mathbf{J}$, and so is a ground-state configuration by our definition. In fact, a stronger statement is possible: barring fine tuning of the couplings, $\mathscr{H} = -\mathbf{J} \cdot \mathbf{s}$ is always-minimized by $\mathbf{s}$ at some vertex $x_{SC}$. That is, the (finite) set of vertices of $P_r^{(k)}$ (which are in one-to-one correspondence with the finite set of SCs of $G_r^{(k)}$) represents *all* possible ground states (by our definition) for any Hamiltonian of the form $H_r^{(k)}$.

The above is mostly a review of previous results, obtained by Teubner,[2] who concentrated on the case $k=2$; here our discussion is generalized to arbitrary $k$. Now we impose some symmetry $X$. We then have ${}^X\mathscr{H} = -\mathbf{J}^{\mathbf{X}} \cdot \mathbf{s} = -\mathbf{J}^X \cdot \mathbf{s}^X$; that is, as the couplings $\mathbf{J}$ are restricted to the $X$-invariant subspace (of dimension $d^X$), in minimizing ${}^X\mathscr{H}$ we only need to consider $X$-invariant correlations. Ground states of ${}^XH_r^{(k)}$ then lie at the vertices of the projection ${}^XP_r^{(k)}$ of the polytope $P_r^{(k)}$ onto the $X$-invariant subspace. Given the convenient correspondence between vertices of $P_r^{(k)}$ and SCs of $G_r^{(k)}$ in the general case, we are motivated to seek a graph ${}^XG_r^{(k)}$ (including an appropriate definition of its SCs) such that the correspondence is restored: that is, we wish to construct a graph ${}^XG_r^{(k)}$ whose SCs may be placed in one-to-one correspondence with the vertices of ${}^XP_r^{(k)}$.

We will then use these tools to answer our question: are there disordered ground states of ${}^XH_r^{(k)}$ by our definition—that is, are there ground states of finite entropy density which do not require fine tuning of the couplings in ${}^XH$ (other than that due to symmetry) to ensure the degeneracy?

We can give the answer in schematic form here. Some vertices of $P_r^{(k)}$ will map to points which are not vertices of ${}^XP_r^{(k)}$ under the projection. These will also not appear as SCs of our ${}^XG_r^{(k)}$, and we need not consider them further. Of the remaining SCs of $G_r^{(k)}$, it is obvious that imposing the symmetry $X$ will enforce the degeneracy of symmetry-related SCs (*pairs* of SCs, for the symmetries we consider) over the entire $X$-invariant subspace. This leads us to consider three possibilities:

1. Some SCs map to themselves under $X$. Thus, when $\mathbf{H}$ (i.e., $\mathbf{J}$) points toward a vertex corresponding to such a SC, there is only one configuration without defects, namely, the periodic repetition of that SC. (Here we can more precisely define a "defect" as a finite string of arcs which do not belong to the set of arcs represented by the SC.) We will call this a "periodic" GS.

2. In some cases, a pair of SCs of $G_r^{(k)}$ (a pair of vertices of $P_r^{(k)}$) map to a single SC (vertex) under the projection. Now consider a domain wall

between the two thermodynamic "phases" represented by the two SCs, and assume that its energy is positive.[5] That is, assume that the domain wall is a defect, because it requires arcs not present in either SC: this holds when the pair do not share any nodes in $G_r^{(k)}$. Then, in the ground state, the density of these domain walls is zero. This means in turn that there are arbitrarily long, periodic sequences of one or the other SC in any ground-state configuration. We will call this a "periodic GS, with spontaneous symmetry breaking" (SSB).

3. Finally, consider case 2 with the one change that the domain wall energy is zero. This in turn requires that the domain wall use no "defect" arcs, i.e., that the two SCs share one or more nodes in $G_r^{(k)}$. (This is termed "zero surface tension" in Teubner.)[2] In this case the ground state is disordered by our definition: there are $2^n$ *undefected* ground-state configurations (with $n \equiv N/p$, $N$ the number of spins and $p$ the length of either SC of the pair), so that the entropy density is $\ln 2/p$ ($p$ is of course finite as long as K and $r$ are both finite). In this case, we call the ground state "disordered"[6] and refer to the pair of SCs as a "D-pair" (where "D" is meant to evoke "degenerate and disordered").

Our search for disordered ground states then becomes a search for possible instances of case 3 above: a search for "D-pairs." In the following, we will construct ${}^XG_r^{(k)}$, and then use it to find the values of $k$, $r$, and $X$ for which there are D-pairs. The results (for $X = S$ and $I$) are shown in Table I, which is the principal result of this paper. A second result is that our construction of ${}^XG_r^{(k)}$ enables an explicit, algorithmic enumeration of all the GSs of a given ${}^XH_r^{(k)}$. Here and elsewhere, by "all ground states" we mean "all the vertices (finite in number) of ${}^XP_r^{(k)}$," or, equivalently, "all the simple cycles of ${}^XG_r^{(k)}$." The latter formulation allows us to view the list of GSs (vertices) as a list of (undefected) *spin configurations*, one (or more) of which is invariably a minimizer of the energy density.

[5] A *negative* domain-wall energy of course leads to a ground state which is a periodic array of domain walls. This periodic ground state is represented by a symmetric and non-degenerate SC, which is distinct from either member of the degenerate pair between which the domain walls are defined. (Cf. Section 7.) The symmetric SC (and hence, the case of negative domain-wall energy) is thus simply an example of our first case (1).

[6] Calculations of diffraction patterns for D-pairs, with random mixtures of the degenerate simple cycles, reveal that in some cases delta-function (Bragg) peaks can still dominate the spectrum. In other cases, the spectrum is pure continuous.[9] Hence it is possible for this kind of disorder to be present, but difficult to detect, in real layered materials. One (perhaps essential) aspect of the calculation which allowed the disordered configurations to have Bragg peaks was the (common in studies of polytypism) assumption that the spins represent *relative* coordinates for adjacent pairs of layers.

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One can of course readily find pairs of SCs in $G_r^{(k)}$ which are symmetry-related (hence degenerate under a symmetric Hamiltonian) and which share one or more nodes. The problem is then to determine which (if any) of these pairs are ground states, by our definition. What we find is that the imposition of symmetry, while nicely enforcing the degeneracy of pairs of cycles over the entire (symmetric) coupling-parameter space, can also—due to the degeneracy of *parts* of the pairs—suggest the "decomposition" of the pair into two or more other cycles, which are not related by symmetry, and one of which *must be lower in energy density than the degenerate pair*. Decomposition in this sense of a degenerate pair of cycles, when it occurs, excludes that pair from our set of GSs of ${}^XH$. That is, an SC of $G_r^{(k)}$ decomposes under the application of $X$ if and only if the corresponding vertex of $P_r^{(k)}$ fails to map to a vertex of ${}^XP_r^{(k)}$ under the projection to the $X$-invariant subspace.

Hence the search for disordered ground states reduces to the problem of finding symmetry-related pairs of cycles which share one or more nodes but do not decompose. (We will give explicit examples of decomposition below.) Our construction of ${}^XG_r^{(k)}$ and our definition of its SCs are designed to solve this problem. Below we show in detail how this is accomplished, for various combinations of spin ($S$) and space ($I$) inversion symmetry.

## 3. SPIN INVERSION (*S*)

By spin inversion for $k$-state problems we mean the following: the states, which we formally label $(0, 1,..., k-1)$, map under $S$ to $(k-1, k-2,..., 0)$. Nodes, arcs, and cycles of the graph $G_r^{(k)}$ also map to their spin-inverses: $\mathcal{N} \to \bar{\mathcal{N}}$, $arc \to \overline{arc}$, and $cyc \to \overline{cyc}$. We recall that the energetics of our discrete problem is reflected in the weights $w$ assigned to the arcs of the graph $G_r^{(k)}$; the symmetry of $H$ is then reflected in $w(arc) = w(\overline{arc})$ and hence $w(cyc) = w(\overline{cyc})$.

The ground states of the symmetric Hamiltonian ${}^SH$ are the vertices of the section[7] of $P_r^{(k)}$ defined by the lower dimensional space which is invariant under $S$; we call this section ${}^SP_r^{(k)}$. The vertices of ${}^SP_r^{(k)}$ correspond either to a symmetric SC or to an $S$-related pair of SCs. In the latter case, if the pair shares one or more nodes, it will give rise to an uncountably infinite set of degenerate GSs, "most" of which are disordered mixtures of the two SCs.

[7] Strictly, the pertinent lower dimensional object is the projection of $P_r^{(k)}$ onto the invariant hypersurface defined by the symmetry. However, one can show, using the symmetry and convexity properties of $P_r^{(k)}$, that the projection is the section.

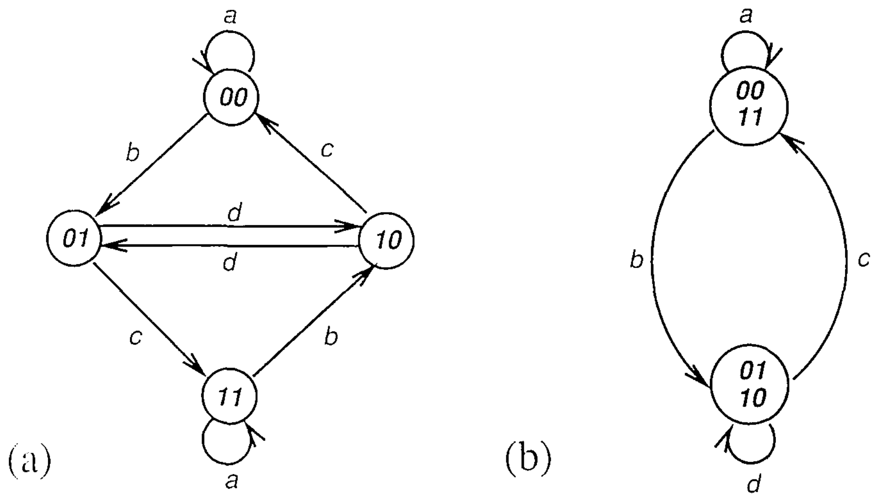



Fig. 1. The graphs (a) $G_2^{(2)}$ and (b) ${}^SG_2^{(2)}$. Arcs which have equal weight by $S$ symmetry are given the same label. Note that the latter graph is isomorphic to $G_1^{(2)}$ (Fig. 8).

In order to discover which of these possibilities can occur for a given $(k, r)$, we would like to find a graph ${}^SG_r^{(k)}$ with the same properties with respect to ${}^SH$ that $G_r^{(k)}$ possesses with respect to $H$: all GSs are SCs, and all SCs are GSs. We consider the following construction[8] (Fig. 1). We identify the equal-weight arcs $arc$ and $\overline{arc}$ of $G_r^{(k)}$ with the single arc $arc^S$ of ${}^SG_r^{(k)}$; similarly, we merge the nodes $\mathcal{N}$ and $\overline{\mathcal{N}}$ to a single node $\mathcal{N}^S$. The resulting graph, for $k=2$, has the nice property (as may be guessed from Teubner)[2] that ${}^SG_r^{(2)} \sim$ (is isomorphic to) $G_{r-1}^{(2)}$. For larger $k$, ${}^SG_r^{(k)}$ is in general no longer a de Bruijn graph, since it includes parallel arcs.

We define an SC of ${}^SG_r^{(k)}$ in precise analogy to an SC of $G_r^{(k)}$: it is a cycle which visits no node in ${}^SG_r^{(k)}$ more than once.

Now we want to show that only SCs of ${}^SG_r^{(k)}$ can be GSs of ${}^SH$. Clearly it is sufficient to restrict our attentions non-SCs of ${}^SG_r^{(k)}$ which are

[8] We are indebted to Jim Hanson for the following observation. In the language of computation theory, the graph $G_r^{(k)}$ is a finite-state transducer whose input is a string of spin values and whose output is a string of arc weights. If we disregard its input (the spins) and consider the possible outputs (the possible strings of arc weights) as the language recognized by the machine, then $G_r^{(k)}$ is a deterministic, finite-state automaton or DFA. A DFA is "minimal" when there is no DFA with fewer nodes which can recognize the same language; and the general $G_r^{(k)}$ is minimal. Given $S$ symmetry, however, $G_r^{(k)}$ is no longer minimal, and can be reduced to a minimal DFA (i.e., one capable of producing the same output strings with fewer nodes) which is in fact ${}^SG_r^{(k)}$ (or equivalent to it). In contrast, $I$ symmetry in general leaves $G_r^{(k)}$ minimal; hence our unconventional approach to ${}^IG_r^{(k)}$ (Section 4). See, for example, Hopcroft and Ullman.[10]

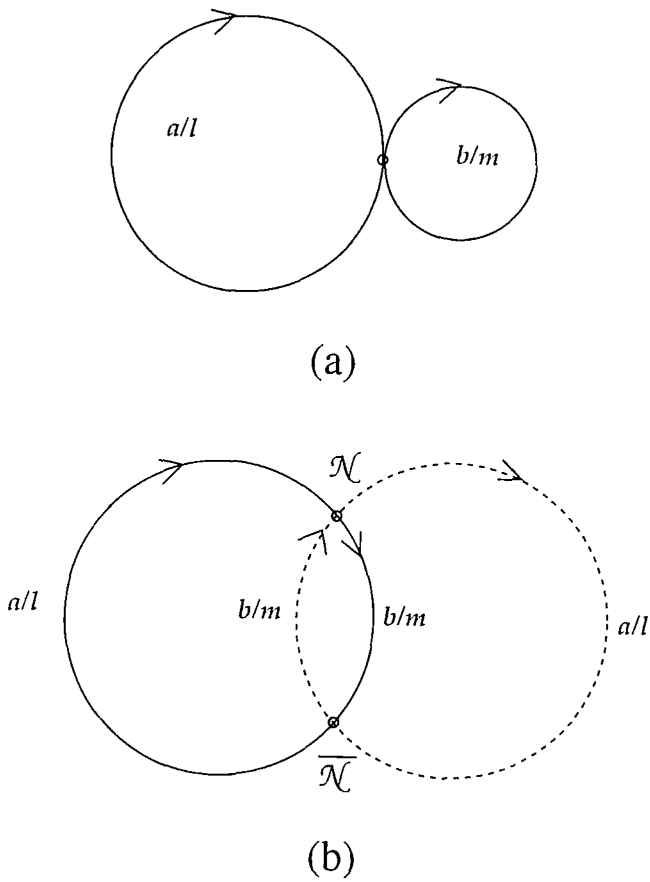



Fig. 2. (a) A (schematic) nonsimple cycle of ${}^S G({}_r^{(k)}$, composed of two simple cycles (SCs) whose per-spin weights are marked. (b) How the non-SC appears in $G_r^{(k)}$: as two, asymmetric but symmetry-related, SCs (one solid, one dashed), each of net weight/spin $(a+b)/(l+m)$. The single shared node in (a) becomes two symmetry-related nodes $\mathcal{N}$ and $\overline{\mathcal{N}}$ in (b). The asymmetric SCs are never ground states of an $S$-symmetric Hamiltonian (see text).

SCs of $G_r^{(k)}$, since we have already ruled out non-SCs of $G_r^{(k)}$. An SC of $G_r^{(k)}$ (visiting no node in $G_r^{(k)}$ twice), which is, however, a non-SC of ${}^S G_r^{(k)}$, will visit at least one node $\mathcal{N}^S$ in ${}^S G_r^{(k)}$ exactly twice (Fig. 2a). In $G_r^{(k)}$, this nonsimple cycle of ${}^S G_r^{(k)}$ represents an SC $cyc$ and its partner $\overline{cyc}$. Schematically (Fig. 2b) we can represent these cycles as

$$cyc: \mathcal{N} \xrightarrow{b/m} \overline{\mathcal{N}} \xrightarrow{a/l} \mathcal{N} \tag{2}$$

$$\overline{cyc}: \overline{\mathcal{N}} \xrightarrow{b/m} \mathcal{N} \xrightarrow{a/l} \overline{\mathcal{N}} \tag{3}$$

Here each arrow represents a path (a composition of arcs), and the energy per spin of each path is placed above the arrow. The energy per spin of *cyc* and $\overline{cyc}$ is then $(a+b)/(l+m)$. It is apparent from Fig. 2b that *cyc* and $\overline{cyc}$ together define two other cycles, one with energy/spin $a/l$, the other with $b/m$. We assume that $a/l < b/m$. We then use the fact that

$$a/l < (a+b)/(l+m) < b/m \tag{4}$$

to deduce that the non-SC $cyc/\overline{cyc}$ of ${}^{S}G_r^{(k)}$ is not a GS of ${}^{S}H$: barring fine tuning of parameters (as would be needed to set $a/l = b/m$), one of the symmetric cycles is always lower in energy density. Alternatively, we say that the non-SC $cyc/\overline{cyc}$ *decomposes* into the two cycles with intensive energy $a/l$ and $b/m$. This conclusion holds without any restriction on the number of nodes which may be shared between the $a/l$ and $b/m$ paths—which therefore also may decompose.

Hence we find that all cycles of ${}^{S}G_r^{(k)}$ which are not SCs of ${}^{S}G_r^{(k)}$ are not GSs of ${}^{S}H$. Now we wish to show that all the SCs of ${}^{S}G_r^{(k)}$ are GSs of ${}^{S}H$. It is helpful to recall the generic case first, since the argument is then readily generalized to the case of $S$ symmetry.

We consider the graph $G_r^{(k)}$ and the corresponding polytope $P_r^{(k)}$. The vertices of $P_r^{(k)}$ represent extrema of the Hamiltonian $\mathscr{H}$, and hence GSs. Since the polytope $P_r^{(k)}$ is defined[2] by the intersection of a set of inequalities (of number $n_f$) on the correlations of $\mathscr{H}$, the faces of $P_r^{(k)}$ are a set of $n_f$ equalities. As shown by Teubner[2] (and as noted above), the (in)equalities for the correlations may be derived from the simpler (in)equalities for the densities $n_\sigma$. The equality which determines each of the $n_f$ faces of $P_r^{(k)}$ is $n_\sigma = 0$. Hence $n_f$ is simply the number of distinct densities, where "distinct" means "not constrained to be equal."

Constraints on the densities arise in the following way. Since the "flow" in the graph $G_r^{(k)}$ is "incompressible," densities for the case where a single arc enters a node and a single arc leaves it are forced to be equal. This flow-induced constraint must be accounted for in order to count correctly the faces of the correlation polytope.[2]

Now consider the restriction to $H = {}^{S}H$, with corresponding polytope ${}^{S}P_r^{(k)}$. A further constraint on the densities arises when we project $P_r^{(k)}$ to the lower dimensional, symmetry-invariant subspace. In this subspace, a configuration and its symmetry-related partner give rise to the same point, since they have the same symmetric correlations. Put more simply, a correlation $\langle \sigma_1^{p_1} \sigma_2^{p_2} \cdots \sigma_q^{p_q} \rangle$ and its symmetry partner (obtained from $\sigma_i \to \overline{\sigma_i}$) represent the same coordinate in the projected subspace. Since the densities represent a linear transformation on the correlations, it follows that the densities $n_\sigma$ and $n_{\bar{\sigma}}$ are also identified in the invariant subspace.

Hence $n_\sigma$ and $n_{\bar\sigma}$ cannot be considered distinct when we work in the invariant subspace.

Now consider an SC of ${}^SG_r^{(k)}$. The argument is essentially the same as for the generic case. The (locus of the) SC lies on the surface of ${}^SP_r^{(k)}$, since the SC cannot visit all the arcs of ${}^SG_r^{(k)}$. And the *distinct* (with the above constraint, $n_\sigma$ not distinct from $n_{\bar\sigma}$) densities of the SC are fixed by the fact that the SC represents an *unambiguous* path in ${}^SG_r^{(k)}$. Finally, varying the densities from those of an SC again requires violating one or more further equalities. Hence, SCs of ${}^SG_r^{(k)}$ are vertices of ${}^SP_r^{(k)}$, and so are GSs of ${}^SH$.

Hence we have shown that the SCs of ${}^SG_r^{(k)}$ are the GSs of ${}^SH$. Our search for disordered GSs now takes the form: when does an SC of ${}^SG_r^{(k)}$ represent a pair of node-sharing SCs of $G_r^{(k)}$—a "D-pair"?

The question in this form can be answered, and the answer is simple. An SC of ${}^SG_r^{(k)}$ represents a D-pair only if it includes an $S$-invariant node $\mathcal{N}^*$. Such a cycle (excluding the "ferromagnetic" SC which uses the arc $\mathcal{N}^* \to \mathcal{N}^*$) maps to a pair of cycles in $G_r^{(k)}$ sharing the node $\mathcal{N}^*$. All other SCs of ${}^SG_r^{(k)}$ map to either a single, symmetric cycle (of twice the period) in $G_r^{(k)}$, or to a pair of SCs sharing no nodes. (As shown above, no SC of ${}^SG_r^{(k)}$ maps to a pair sharing two or more nodes.)

The graph $G_r^{(k)}$ contains no $S$-invariant nodes for $k$ even, and one such node [consisting of $r$ consecutive occurrences of the invariant spin value $\sigma^* = (k-1)/2$] for odd $k$. Furthermore, given odd $k$, there is always (i.e., for any $r$) at least one SC of ${}^SG_r^{(k)}$ which uses the node $\mathcal{N}^*$. Hence we conclude the following:

*For even k, S symmetry never gives rise to disordered GSs; for odd k and for every r, S symmetry does give rise to disordered GSs.*

## 4. SPACE INVERSION (*I*)

### 4.1. Preliminaries

The arguments and conclusions with respect to $I$ symmetry are somewhat more involved than those for spin inversion. However, the basic outline of the argument is the same: we wish to define a graph ${}^IG_r^{(k)}$ whose SCs are the GSs of ${}^IH$ (that is, the Hamiltonian $H$ constrained to be invariant under $I$). We will then seek D-pairs among the SCs of ${}^IG_r^{(k)}$. The definition of ${}^IG_r^{(k)}$ and of its SCs will be developed in this subsection, along with a number of auxiliary concepts which are useful for the argument. We follow this subsection with a search (Section 4.2) for D-pairs, arising from $I$ symmetry, in $k$-state problems with $k > 2$. Finally, in the last subsection (Section 4.3), we treat the special (Ising) case $k = 2$.

We encourage the reader who is not interested in technicalities to skip this subsection—at least at first reading. Our conclusions, and the flavor of the argument, may be gleaned from reading Sections 4.2 and 4.3 and referring to Figs. 3–5. We include this subsection here because it defines and explains a number of concepts and terms which are needed for the complete argument and which appear in the subsequent subsections. This subsection is itself divided into three parts. Concepts and terminology are developed in Section 4.1.1, immediately following this paragraph. In Section 4.1.2 we define the graph ${}^{I}G_r^{(k)}$ and its SCs; finally, in Section 4.1.3 we show that these SCs correspond to the GSs of ${}^{I}H$.

**4.1.1. Classification Schemes and Representations.** We begin by developing a classification scheme for the nodes and arcs of $G_r^{(k)}$ according to their behavior under $I$. All the nodes of $G_r^{(k)}$ may be classified, according to their behavior under $I$, into three sets: a "left-handed" set given the label $L$, their space inverses $R$, and an invariant or symmetric set labelled with $S$. Since the arcs of $G_r^{(k)}$ are in one-to-one correspondence with the nodes of $G_{r+1}^{(k)}$, the same holds true for the arcs.

Clearly there are, in general, many ways of choosing the $L$ and $R$ sets. Each such choice can be taken as a constraint on how the graph $G_r^{(k)}$ is to be represented in a planar drawing (e.g., $R$ nodes on the right, $L$ nodes on the left). We are of course most interested in those properties of $G_r^{(k)}$ and of ${}^{I}G_r^{(k)}$ which are independent of the choice of representation. However, we will find two types of representation (or rep, for brevity) to be most convenient.

First we define the "recursive" representations. (For $k=2$ and $r \leqslant 5$ there are two; in general, there are many.) In these representations the handedness of all nodes and arcs of $G_r^{(k)}$ are determined, as much as possible, from the handedness of the arcs of $G_r^{(k)}$. Hence, one must first choose a classification for the arcs of $G_1^{(k)}$. (The nodes, of length $r=1$, are all inversion invariant and so $S$.)

One then exploits a representation-independent procedure[6] for constructing $G_{r+1}^{(k)}$ from $G_r^{(k)}$. [Such a procedure gives (by recursion) $G_r^{(k)}$, for any $r$, from $G_r^{(k)}$]. The arcs of $G_r^{(k)}$, representing all the possible $k^{r+1}$ sets of $r+1$ spins, become the nodes of $G_{r+1}^{(k)}$. Arcs of $G_{r+1}^{(k)}$ then represent adjacent sets of $r+2$ spins, which may be traced back to adjacent *pairs* of arcs in $G_r^{(k)}$ (where "adjacent" means meeting at a node, with one arc of the pair incoming, the other outgoing).

Our recursive reps then use the following rules. Since nodes of $G_{r+1}^{(k)}$ come from arcs of $G_r^{(k)}$, we carry the handedness through unchanged. The handedness of the arcs of $G_{r+1}^{(k)}$ is then determined from that of the corresponding arc pairs of $G_r^{(k)}$ as follows:

$$RR \to R \tag{5a}$$

$$SR \text{ or } RS \to R \tag{5b}$$

$$LL \to L \tag{5c}$$

$$SL \text{ or } LS \to L \tag{5d}$$

$$SS \to S,\ L, \text{ or } R\ (= \text{inverse of intervening node}) \tag{5e}$$

$$RL \text{ or } LR \to S\ (\text{if } R \text{ and } L \text{ are related by } I) \tag{5f}$$

$$RL \text{ or } LR \to J\ (\text{otherwise}) \tag{5g}$$

The last line of course needs clarification. We use the symbol $J$ to mean a "joining arc" (JA): an arc that joins an $R$ node to an $L$ node, but is not symmetric itself. We note that symmetry, or lack of same, is rep-independent; however, whether or not a given arc is $J$ is rep-dependent, since one can always move one of the nodes. Joining arcs will be significant in many parts of our discussion of $I$ symmetry. Specifically, the proper treatment of JAs is an essential part of our definition of ${}^{I}G_r^{(k)}$ and its SCs; also, the concept of JAs is the simplest way to understand our results for $k=2$, for which we will distinguish the cases $r<5$ from those with $r \geqslant 5$. Note that, for the purpose of proceeding with a recursive rep, the handedness of a JA must be chosen arbitrarily to be $L$ or $R$.

This completes our description of recursive reps. We now define a second type of representation, a "minimal" rep, as follows. In a minimal rep the number $n_J$ of JAs is minimized. That is, all minimal reps have the same $n_J$, and every rep that is not minimal has a larger $n_J$.

We can always draw the graph $G_r^{(k)}$ in the plane such that it is reflection symmetric about a line $\mathscr{I}$ (the reflection accomplishing $L \leftrightarrow R$ for both nodes and arcs). If there are no JAs, the line $\mathscr{I}$ then partitions $G_r^{(k)}$, drawn in this way, into disjoint sets of arcs and nodes, with $R$ and $L$ arcs/nodes on opposite sides of $\mathscr{I}$, and only $S$ arcs/nodes touching $\mathscr{I}$. Thus if $n_J=0$ (in any rep), we call the graph "$\mathscr{I}$-disjoint." If $n_J \neq 0$ in a minimal rep, the graph is non-$\mathscr{I}$-disjoint. Obviously, $\mathscr{I}$-disjointness is a property which is independent of rep, but most easily ascertained in a minimal rep.

Not all minimal reps are recursive. For example, for the case $k=2$, and in the absence of JAs, there are only two recursive reps. These two reps are trivially related, since there are only two arcs in $G_1^{(2)}$ (cf. Fig. 8) which are not $S$—and they must have opposite handedness. Since the relation of the two is trivial, then, as long as JAs do not arise [and so introduce ambiguity in going from $(r-1)$ to $r$], we can refer to a "single" recursive rep for $k=2$. We find, by construction, for $k=2$ that "the" recursive rep is also minimal for all $r \leqslant 5$. [This is the reason for the rule for $SS$

combinations, Eq. 5e.] However, for $r=6$ there are multiple recursive reps due to the appearance of JAs at $r=5$; and no recursive rep is minimal. These results, besides demonstrating that minimal reps are not in general recursive, will be useful in our discussion of the $k=2$ case below.

**4.1.2. The Graph $^{I}G$ and Its SCs.** We now seek the graph $^{I}G_r^{(k)}$, which will serve the same purpose, in the case of $I$ symmetry, as was served by $^{S}G_r^{(k)}$ for $S$ symmetry. Unfortunately, it is impossible (see footnote 8) to draw consistently a graph strictly analogous to $^{S}G_r^{(k)}$, that is, a graph in which arcs and nodes related by $I$ are identified.

We can however construct $^{I}G_r^{(k)}$ by broadening our notion of a graph (and of a cycle). We draw $G_r^{(k)}$ to be reflection-symmetric about the symmetry line $\mathscr{I}$ as described above, with $L$ arcs and nodes to the left, $R$ to the right, $S$ nodes on $\mathscr{I}$, and $S$ (and $J$) arcs crossing $\mathscr{I}$. We then construct $^{I}G_r^{(k)}$ by simply erasing everything to one side of $\mathscr{I}$ (Fig. 3). The resulting "graph" (we will drop the quotes) has the odd property that some arcs begin and end on $\mathscr{I}$, rather than on a node. We define a "cycle" of $^{I}G_r^{(k)}$ to be one of two types: (i) a closed path as in a conventional digraph, or (ii) a path which begins and ends on $\mathscr{I}$. A cycle of type (i) will map to two distinct ($I$-related) cycles in $G_r^{(k)}$. A type (ii) cycle becomes a cycle in $G_r^{(k)}$ by simple reflection about $\mathscr{I}$.

Our prescription for $^{I}G_r^{(k)}$ is still not complete; joining arcs in $G_r^{(k)}$ require special handling. Assume the JA *arc* connects nodes $\mathscr{N}_L \to \mathscr{N}_R$ in $G_r^{(k)}$, that $\mathscr{N}_R \neq \bar{\mathscr{N}}_L$ (where $\bar{\mathscr{N}}$ is the spatial inverse of the node $\mathscr{N}$), and that we want to build $^{I}G_r^{(k)}$ by erasing the right half of $G_r^{(k)}$. We then represent *arc* by drawing a heavy line (to distinguish the JA from the non-JAs) from $N_L$ to $\bar{\mathscr{N}}_R$ (which is in the left half). Furthermore, in $^{I}G_r^{(k)}$, *arc* is a "sink": it *leaves both* $N_L$ and $\bar{\mathscr{N}}_R$. (Had *arc entered* $N_L$, it would be a "source": it would also enter $\bar{\mathscr{N}}_R$.) The resulting construction is shown in Fig. 4, using $G_5^{(2)}$ as an example.

This new feature of $^{I}G_r^{(k)}$ requires yet further broadening of our definition of a cycle. The rule is that two paths flowing (in the same direction) from a single source to a single sink also constitute a cycle. We also allow the possibility that $\mathscr{I}$ can act as a source or sink. For bookkeeping, we label the paths leaving a source (and entering a sink) with distinct "colors" (hence only two colors are needed). We allow further sources and/or sinks (i.e., JAs) in each path—with the path changing color (and apparent direction) when crossing a JA—with the constraint that the colors must match at every node that is neither source nor sink.

Clearly, a type (i) (closed in $^{I}G_r^{(k)}$) cycle which does not touch $\mathscr{I}$ must then include an even number of JAs, so that the path directions (colors) match everywhere away from the JAs. A type (ii) can include an odd number, in which case it uses $\mathscr{I}$ either as a source or as a sink.

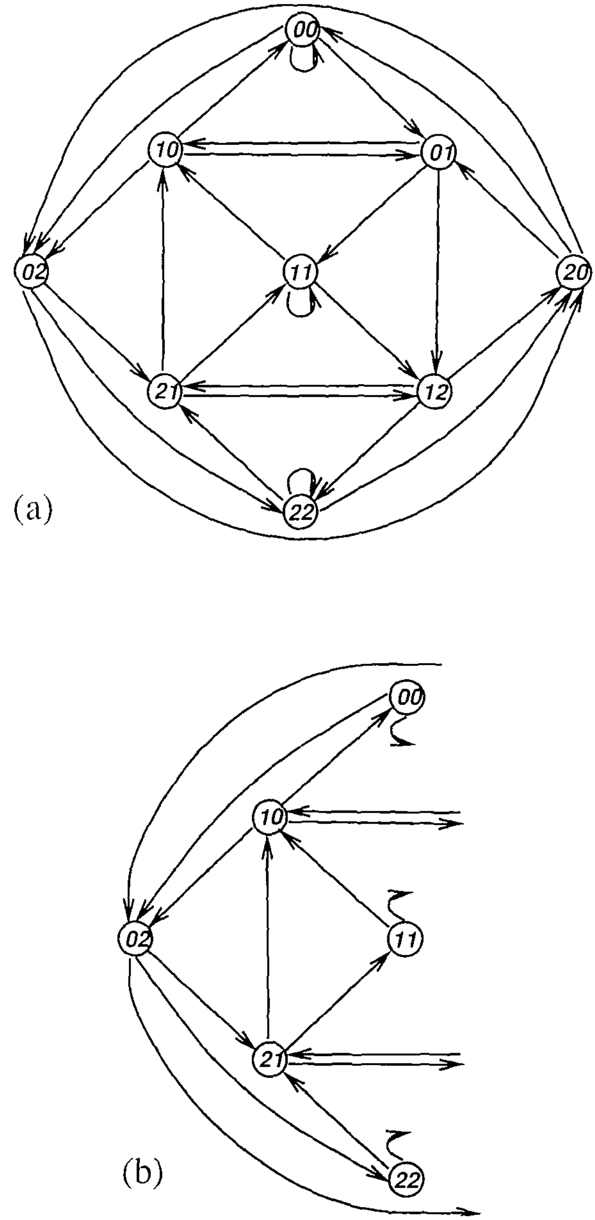



Fig. 3. (a) $G_2^{(3)}$; (b) ${}'G_2^{(3)}$. The latter is obtained from the former by removing everything to the right of a vertical line of symmetry ($\mathscr{I}$) in $G_r^{(k)}$; arcs can then begin and end on this line, as seen in (b). See the text for a definition of cycles and simple cycles of ${}'G_r^{(k)}$.

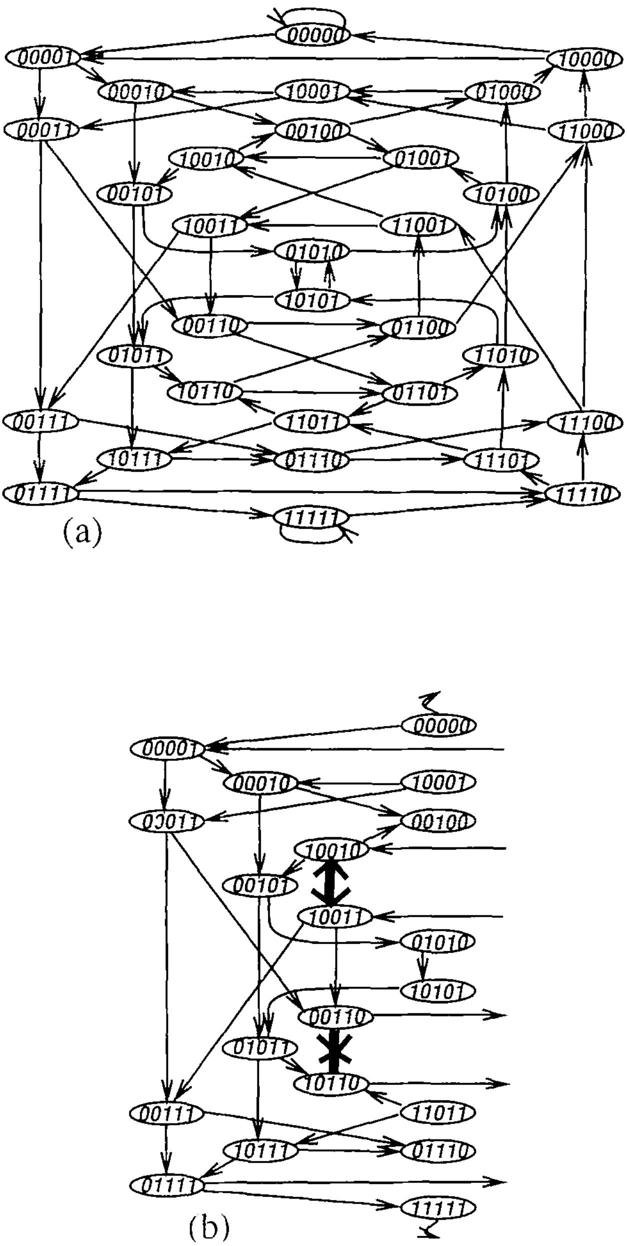
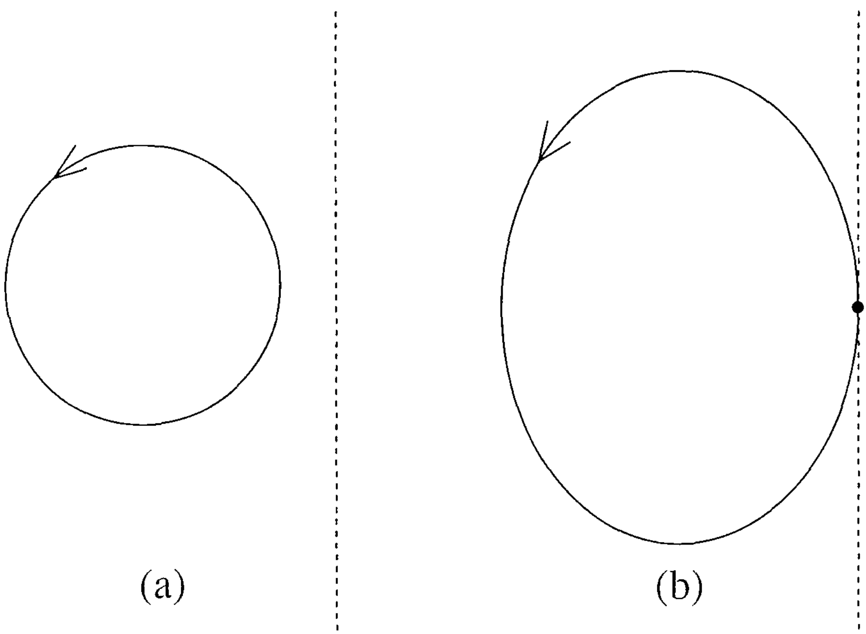



Fig. 4. (a) $G_5^{(2)}$; (b) $'G_5^{(2)}$. Note the appearance of the joining arcs (JAs—asymmetric arcs crossing the central line of symmetry) in both (a) and (b); in the latter, JAs appear as heavy lines with double arrowheads.

Since sources and sinks in ${}^{I}G_r^{(k)}$ merely amount to crossing $\mathscr{I}$ in $G_r^{(k)}$, it is perhaps clear that our specification of cycles of ${}^{I}G_r^{(k)}$ will yield, upon "unfolding" to $G_r^{(k)}$, cycles of the latter as well.

This completes our construction of ${}^{I}G_r^{(k)}$, including the definition of its cycles. However, we still need an appropriate definition of a *simple* cycle of the graph ${}^{I}G_r^{(k)}$. We will classify the SCs of ${}^{I}G_r^{(k)}$ into four topological types. First we consider those that use no sources or sinks, i.e., a single color. Such SCs include (1) a simple closed loop in ${}^{I}G_r^{(k)}$, which does not touch $\mathscr{I}$ (Fig. 5a); (2) the same as (1), except one node (and only one) is on $\mathscr{I}$ (Fig. 5b); and (3) a non-self-intersecting path from one node on $\mathscr{I}$ to another (Fig. 5c).

We now consider SCs using one or more sources/sinks. We can add an even number of JAs to a type (1), and will include the resulting SCs in



(a) (b)

Fig. 5. The SCs of ${}^{I}G_r^{(k)}$, presented in schematic form as four topological types. In each case the symmetry line $\mathscr{I}$ is marked by a vertical dashed line. (a) A type (1) SC. (b) A type (2) touches $\mathscr{I}$ at a single node. (c) A type (3). (d) A type (4) SC. This is an example of an SC consisting of two paths ("red" and "blue") running from source to sink. The red and blue touch at one or more contiguous nodes (an RBC, marked by a single large dot) in the center. (e) A type (2) which uses $\mathscr{I}$ as a source and has an RBC (the upper heavy line) which includes $\mathscr{I}$.

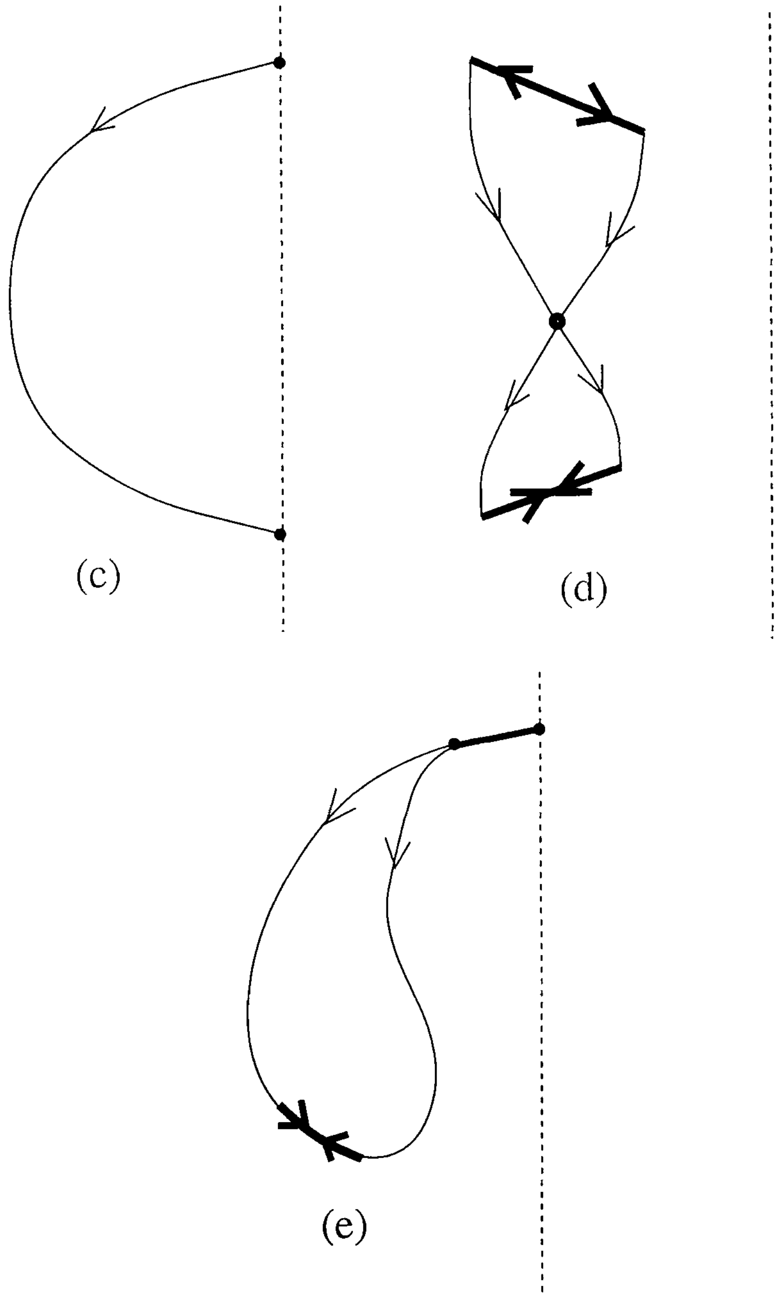



Fig. 5 (*continued*)

type (1). A type (2) can add an even number of JAs by having only one color at $\mathscr{I}$, or an odd number by having $\mathscr{I}$ serve as source or sink. The same holds true for type (3).

The final step is to consider allowing paths of differing color to touch. Such a rule makes sense, since, when unfolded, such paths are on opposite sides of $G_r^{(k)}$ and so do not intersect in $G_r^{(k)}$. Assume two paths of different color ("red" and "blue") touch in a contiguous sequence of nodes. Call this part a "red/blue contact" or RBC. We allow only a single RBC in an SC of ${}^I G_r^{(k)}$; furthermore, there are constraints on how the paths terminate (meet) at each end. The red and blue paths may diverge before annihilating at a JA in ${}^I G_r^{(k)}$—but, once diverged, may not recontact, by the "one-RBC" rule. In contrast, if, at either end, the two annihilate at $\mathscr{I}$, then *no* separation of the two is allowed; that is, the RBC must include $\mathscr{I}$.

How do these new possibilities alter our topological types? We add a single RBC to type (1), "pinching" it somewhere such that red meets blue. We call the result type (4); it is an RBC terminated by JAs at both ends (Fig. 5d). By the above rules, we can only pinch a type (2) starting from $\mathscr{I}$; the result.is still a type (2), but with an RBC "neck" of more than one node (Fig. 5e). Finally, we can either pinch a type (3) not at all, or everywhere—however, a type (3) that is all RBC is equivalent to one which has no RBC. Thus, the addition of RBCs augments our list of topological types of SCs by one.

An important feature of our types (1)–(4) is that they are independent of the representation chosen for the graphs $G_r^{(k)}$ and ${}^I G_r^{(k)}$. Changing the rep amounts to exchanging the handedness of pairs of nodes, and allowing the arcs to follow. For any given SC, the effect of changing the rep is to change the number of JAs in the SC by an even integer. One simple way to see that our types are invariant under such changes is to unfold each one into the full graph $G_r^{(k)}$. The four types then appear as (1) two nonintersecting loops (2) two loops joined at one node (or series of contiguous nodes) (3) a single loop, and (4) a structure which may be viewed as two loops, joined at two distinct nodes (or series of nodes). Changing reps then amounts to moving nodes (i.e., across $\mathscr{I}$)—a process which cannot change the topology, either in $G_r^{(k)}$ or in ${}^I G_r^{(k)}$.

Our specification of SCs of ${}^I G_r^{(k)}$ is considerably more involved than that for ${}^S G_r^{(k)}$. This is because of two complications: the treatment of $\mathscr{I}$ as a node, and the use of two colors, with the consequent possibility of RBCs. However, our overall criterion for an SC of ${}^I G_r^{(k)}$ is the same as that for an SC of ${}^S G_r^{(k)}$ or of $G_r^{(k)}$: *an SC of* ${}^I G_r^{(k)}$ *is one for which there is no ambiguity as to which arcs of* ${}^I G_r^{(k)}$ *are to be traversed.* As we will see, it is this "no-ambiguity rule" (implicit in the above detailed rules) which prevents SCs of ${}^I G_r^{(k)}$ from decomposing.

Our definition of ${}^{I}G_r^{(k)}$ and its SCs is complete. We now proceed to show that the SCs of ${}^{I}G_r^{(k)}$ are the GSs of ${}^{I}H$.

**4.1.3. SCs of ${}^{I}G$ = GSs of ${}^{I}H$.** First we show that non-SCs of ${}^{I}G_r^{(k)}$ are not GSs of ${}^{I}H$. Our specification of SCs of ${}^{I}G_r^{(k)}$ as a list of possibilities amounts to forbidding three things: (i) self-intersection with the same color, (ii) more than one RBC, or (iii) improper termination of a single RBC. We next consider violating these three prohibitions, in order. Our goal is to show that violation of any of (i)–(iii) means the resulting cycle is not a GS of ${}^{I}H$.

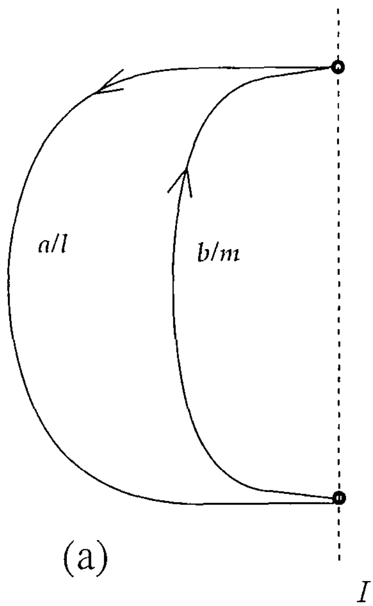
(i) Assume self-intersection by paths of the same color in a cycle of ${}^{I}G_r^{(k)}$. We recall that cycles of ${}^{I}G_r^{(k)}$ in general give rise to pairs of cycles of $G_r^{(k)}$, with the two members of the pair related by $I$. Clearly, a necessary condition for a cycle of ${}^{I}G_r^{(k)}$ to be a GS of ${}^{I}H$ is that the resulting pair in $G_r^{(k)}$ be a pair of SCs (related by $I$). The pair must also not decompose, in the sense described in Section 3 (on spin inversion symmetry).

Most cases of violation of (i) will give pairs in $G$ which are not SCs and so fail the first test. An exception is a same-color contact occurring on $\mathscr{I}$, as shown in Fig. 6a. (This represents a same-color contact because $\mathscr{I}$ is a single "node," which here is visited twice.) This type of cycle unfolds in $G$ as shown in Fig. 6b, and decomposes into two symmetric cycles in a manner similar to that seen in the case of $S$ symmetry. A variation on this violation, which also represents (and also decomposes to) a pair of type (3) SCs in ${}^{I}G_r^{(k)}$, is shown schematically in Figs. 6c and 6d.

(ii) Consider a cycle of ${}^{I}G_r^{(k)}$ with two RBCs. We suppose that the two RBCs, which are connected by a red/blue "bubble," are terminated by a JA at one end (a "stirrup"—cf. Fig. 7a) and by $\mathscr{I}$ at the other. The resulting set of arcs in $G$ (Fig. 7b) may be viewed, with some care, as a symmetry-related pair of SCs, of intensive energy $(a+b+c)/(l+m+n)$. However, because the no-ambiguity rule is violated, this pair also defines two other (symmetric) SCs of (intensive) weight $(2a+c)/(2l+n)$ and $(2b+c)/(2m+n)$, respectively. Now assume the pair represented by Fig. 7a is a GS, so that

$$\frac{a+b+c}{l+m+n} < \frac{2a+c}{2l+n}, \qquad \frac{a+b+c}{l+m+n} < \frac{2b+c}{2m+n} \tag{6}$$

If we multiply out each of these inequalities (assuming $l$, $m$, and $n$ positive), we obtain a contradiction. Hence the non-SC in Fig. 7a cannot be a GS; it decomposes (Fig. 7c and 7d). The same arithmetic and conclusion, with slightly different pictures, apply when the pair of RBCs is terminated by



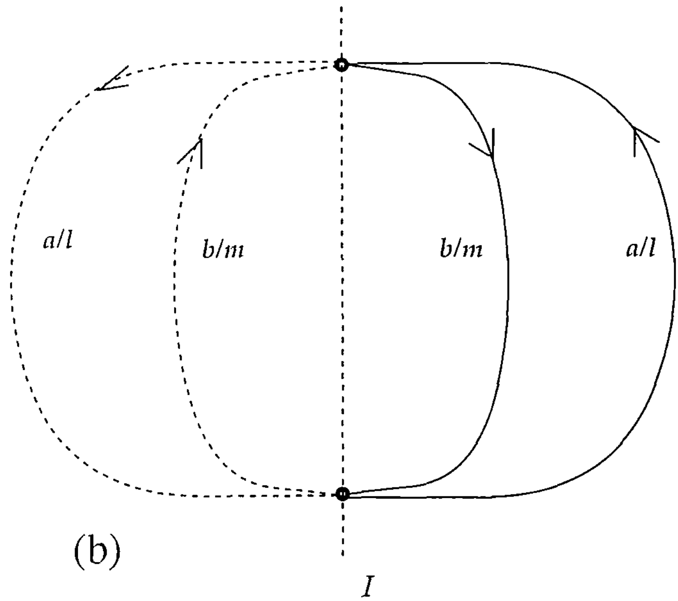
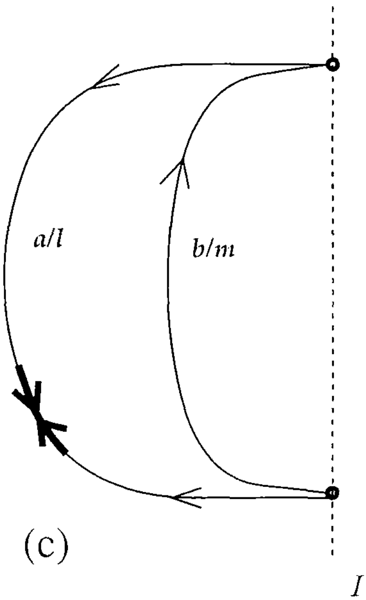




Fig. 6. (a) A non-SC of ${}^{I}G_r^{(k)}$ (schematic). (b) The appearance of (a) in $G_r^{(k)}$; compare Fig. 2b. Again the asymmetric SCs, of weight $(a+b)/(l+m)$, are never ground states when $H$ is I-symmetric. Note that changing the relative direction of the two pieces in (a) [or in (c)] leaves the logic unchanged. (c, d): Same as (a, b), except there is a JA in part of the cycle.



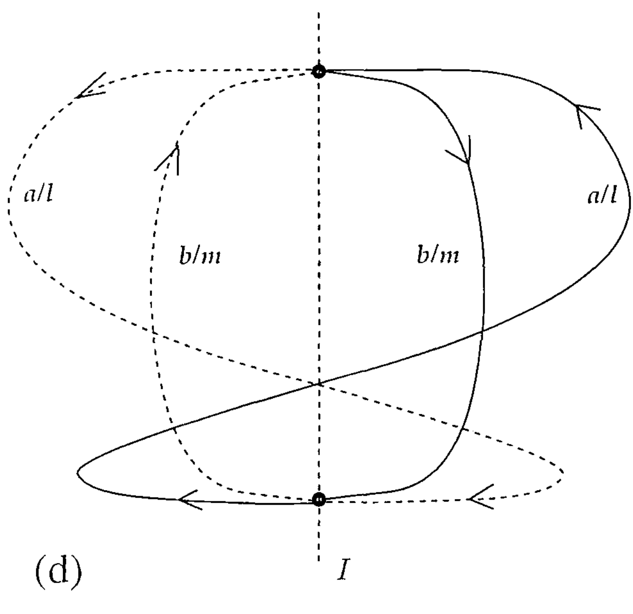




Fig. 6 (*continued*)

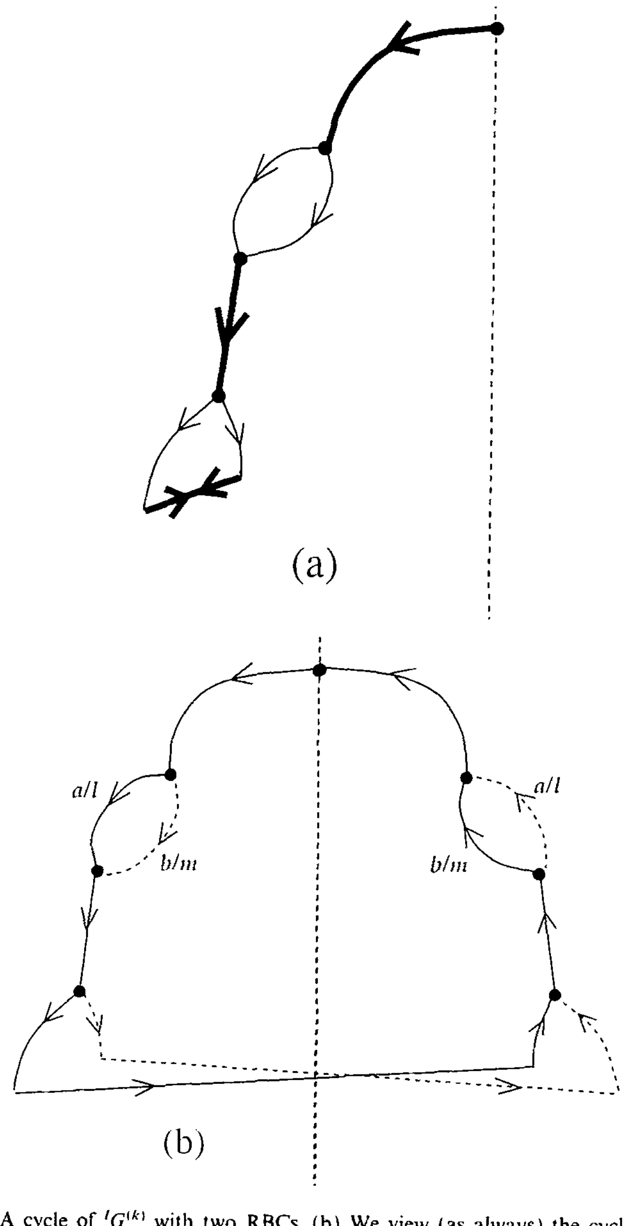



Fig. 7. (a) A cycle of ${}'G_r^{(k)}$ with two RBCs. (b) We view (as always) the cycle of ${}'G_r^{(k)}$ [in (a)] as representing two symmetry-related cycles in $G_r^{(k)}$, shown here. Where the two cycles differ, one is dashed and the other solid. We take the net intensive weight of all arcs outside the "bubbles" to be $c/n$—which is the same for both cycles by symmetry. Each of the two cycles is an SC of $G_r^{(k)}$ but neither is a GS of ${}'H$ (see text). (c, d) The decomposition of the cycle of (a) to two [type (2)] SCs of ${}'G_r^{(k)}$, each with a single long RBC. Both (c) and (d) represent GSs of ${}'H$.

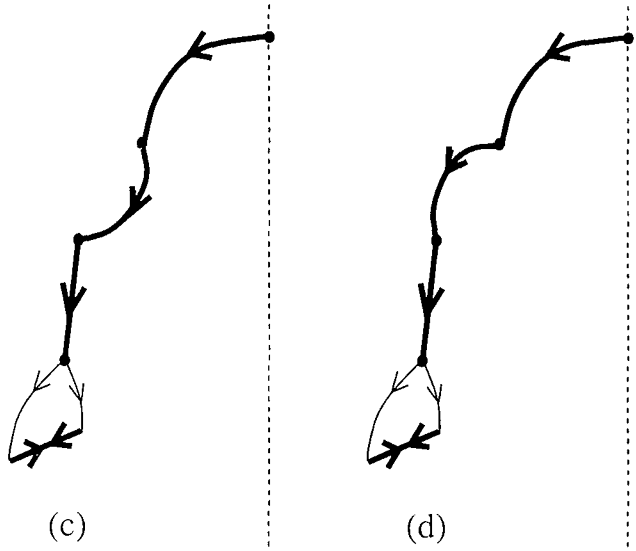



Fig. 7 (*continued*)

stirrup + stirrup, or by $\mathscr{I} + \mathscr{I}$. Therefore, any SC of $G_r^{(k)}$ which is not an SC of ${}^{I}G_r^{(k)}$ by virtue of violation of (ii) is not a GS of ${}^{I}H$.

(iii) We finally consider improper termination of an RBC, by divergence before termination on $\mathscr{I}$. It is easily seen by simple sketches that, regardless of the other termination of the RBC (stirrup, or properly on $\mathscr{I}$), the resulting pair are not SCs of $G$ and so fail to be GSs of ${}^{I}H$.

Summarizing the above, we find that, for any conceivable violation of the rules for an SC of ${}^{I}G_r^{(k)}$, the resulting cycle is not a GS of ${}^{I}H$. Turning this around, we find that all GSs of ${}^{I}H$ are SCs of ${}^{I}G$.

We note in passing that Morita[11] previously obtained results which foreshadow ours. Morita assumed $I$ symmetry of $H$, and found necessary conditions for cycles of $G$ to be GSs of ${}^{I}H$. In our language, Morita found that such allowed cycles must touch $\mathscr{I}$ no times [types (1) and (4)], or once [type (2)], or, if twice, must be $I$-symmetric [type (3)]. These conditions, which are satisfied by our SCs, are not sufficient to ensure a GS. Specifically, Morita's rules encapsulate our rule (i) (no same-color contact), but fail to capture (ii) and (iii) (which give constraints on RBCs).

Finally, we need the converse: that all SCs of ${}^{I}G_r^{(k)}$ are GSs of ${}^{I}H$. Here the argument is essentially unchanged from that for the case of $S$ symmetry: densities (arcs) related by $I$ are considered not distinct, and it readily follows that SCs of ${}^{I}G_r^{(k)}$ are vertices of ${}^{I}P_r^{(k)}$—the intersection (see footnote 7) of $P_r^{(k)}$ with the $I$-invariant subspace.

### 4.2. $k>2$

We are now ready to seek D-pairs in ${}^{I}G_r^{(k)}$. To this end we can use our established classification of the SCs of ${}^{I}G_r^{(k)}$ into four types. A type (1) SC (Fig. 5a) gives rise to a pair of nonintersecting SCs in $G$. Such a GS thus represents spontaneous breaking of $I$ symmetry (SSB), but not a D-pair. All type (2) SCs represent D-pairs in $G_r^{(k)}$, sharing one (Fig. 5b) or more (Fig. 5e) nodes which straddle the symmetry line $\mathscr{I}$. Type (3) SCs of ${}^{I}G$ (Fig. 5c) map to symmetric SCs of $G$, and so are not D-pairs. Finally, type (4) SCs (Fig. 5d) give D-pairs sharing one or more nodes which do not touch $\mathscr{I}$.

Hence our search for D-pairs is a search for SCs of type (2) or (4) in ${}^{I}G$. We first consider $r=1$. For this case, all nodes are on $\mathscr{I}$. Hence all SCs of ${}^{I}G_1^{(k)}$ are of type (3), and there are no D-pairs.

We now use the fact that (in this subsection) $k>2$, and consider $r>1$, in a recursive rep. In this case, there are always SCs of $G$ whose nodes are of the form $L...LS$ (where ... is a string of $L$'s). Specifically, a cycle of the form $L^jS$ (nodes) in $G_r^{(k)}$ may be built, in a recursive rep, from a cycle in $G_1^{(k)}$ whose arcs (all nodes being $S$) take the form $L^{j-(r-2)}S^{r-1}$. The $I$-partner of such an SC in $G_r^{(k)}$ is then of the form $R^jS$. A pair of SCs of $G$ of this form is a type (2) SC of ${}^{I}G$, with a single node (and a single color) at $\mathscr{I}$, and the pair is a D-pair. For example, in $G_2^{(3)}$, one can build $LLLS$ (nodes) SCs from $LLLS$ (arcs) of $G_1^{(3)}$; these form D-pairs in $G_2^{(3)}$ with their partners $RRRS$. There are three such pairs—(2210)/(0122), (2110)/(0112), and (2100)/(0012)—as may be seen in Fig. 3b. For larger $k$ and/or $r$, the number of such D-pairs increases (and other types appear). Hence we find that:

*Disordered GSs occur in the case of I symmetry for any k-state problem with* $k\geqslant 3$ *and* $r\geqslant 2$.

### 4.3. $k=2$

The Ising case ($k=2$) has some special properties. In particular, $G_1^{(2)}$ (Fig. 8) has a single $L$ arc and a single $R$ arc. This makes it impossible to make any cycle in $G_1^{(2)}$ in which an $L$ arc follows another $L$ (similarly with $R$), so that our above argument by construction, using a recursive rep for $k>2$, fails for $k=2$. In fact, even with multiple $S$ arcs [(11) or (00)] after a given $L$, it is clear that the first non-$S$ arc after the $L$ must be an $R$. In other words, *all* cycles of $G_1^{(2)}$, of any length, are of the form

$$\ldots R\ldots L\ldots R\ldots L\ldots R\ldots L\ldots \tag{7}$$

where ... is a string of $S$'s of any length.

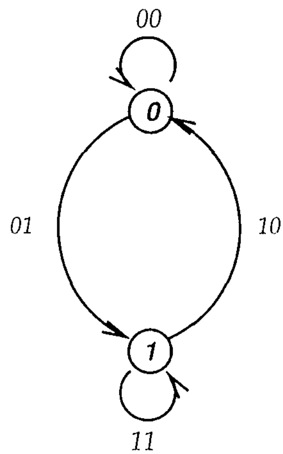



Fig. 8. $G_1^{(2)}$. The choice of handedness for the two nonsymmetric arcs of $G_1^{(k)}$ is really no choice, since the two must be opposite. Hence, until JAs occur at $r=5$, there is effectively only one recursive rep for $k=2$.

From this we can deduce another property which is peculiar to the Ising case in the recursive rep. The property (7) implies that every cycle of $G_1^{(2)}$ [except the two ferromagnetic cycles (1) and (0)] crosses $\mathscr{I}$ an even ($\geqslant 2$) number of times. One can easily show that the same is true for $G_r^{(2)}$, for any $r \leqslant 5$, in the recursive rep.[9] We then note the following: (i) the recursive rep is a minimal rep for $r \leqslant 5$; (ii) in the recursive rep, $G_r^{(2)}$ has $n_J=0$ for $r<5$ and $n_J>0$ for $r \geqslant 5$; and (iii) hence $G_r^{(2)}$ is $\mathscr{I}$-disjoint for $r<5$ and non-$\mathscr{I}$-disjoint for $r=5$. (These statements may be verified by explicit construction of the recursive rep.)

Our "even-crossing rule", plus (i)–(iii) of the previous paragraph, suffice to ensure that there are neither D-pairs nor SSB (of $I$) in $G_r^{(2)}$ for $r<5$, as follows: Type (1) cannot occur in the recursive rep due to even-crossing plus $\mathscr{I}$-disjointness; hence it cannot occur in any rep. Type (2) is similarly ruled out. Type (4) requires JAs; this type is ruled out by (iii) for $r<5$. Thus all SCs·of ${}^I G_r^{(2)}$, for $r<5$, are of type (3)—symmetric cycles of G —and hence represent neither D-pairs nor SSB.

We next consider the case $r=5$. Here we find four JAs (Fig. 4) in the recursive rep (which is still minimal). The presence of these JAs is sufficient

[9] For $r>5$ there are multiple recursive reps (due to the appearance of JAs at $r=5$). We conjecture (but do not need for our argument) that the even-crossing property holds in all recursive reps, even for $r>5$.

to allow three of the four types of SCs in ${}^{I}G_5^{(2)}$. For example, a type (2) SC and its unfolding are shown in Fig. 9. An example (the only one) of SSB in $G_5^{(2)}$ is the pair (101100) and its inverse (001101); this pair gives a type (1) SC of ${}^{I}G_5^{(2)}$ which uses both of its JAs.

Given that $G_r^{(2)}$ is non-$\mathscr{I}$-disjoint, one can show that $G_{r+1}^{(2)}$ is also. (In fact, this is true for any $k$.[10]) Furthermore, even in a minimal rep, $n_J$ increases with increasing $r$. The result is that types (1)–(3) occur for all $r \geqslant 5$. We also find (aided by a computer search) that type (4) D-pairs[11] occur for $r \geqslant 7$. Hence we find that:

*Both SSB and disordered, degenerate GSs occur in the Ising problem with I symmetry for $r \geqslant 5$. Neither occurs for $r < 5$.*

## 5. COMBINED SYMMETRIES

### 5.1. *SI* Symmetry

*SI* symmetry may be handled very much like $I$. (The symmetry line in this case is a horizontal line $\mathscr{S}\mathscr{I}$ through the center of the graphs.) Here we just note the conclusions. The results for $k = 2$ are the same (an "even-crossing rule" in the recursive rep, non-$\mathscr{S}\mathscr{I}$-disjoint for $r \geqslant 5$, and disordered GSs for $r \geqslant 5$). $G_r^{(3)}$ is non-$\mathscr{S}\mathscr{I}$-disjoint for $r \geqslant 3$; $G_r^{(k)}$ is non-$\mathscr{S}\mathscr{I}$-disjoint for any $k \geqslant 4$ and $r \geqslant 1$. There are disordered GSs for any $k \geqslant 3$ and $r \geqslant 1$.

### 5.2. $(S+I)$ Symmetry

We finally consider the case where both $S$ and $I$ are good symmetries of $H$. We construct the graph ${}^{S+I}G_r^{(k)}$ by applying $I$ symmetry to ${}^{S}G_r^{(k)}$ (erasing half of it, and correcting for JAs). We can then use arguments like those above to show that the SCs of ${}^{S+I}G_r^{(k)}$ (defined similarly to those of ${}^{I}G_r^{(k)}$) are the GSs of ${}^{S+I}H$. (In particular, the same arguments used in Section 4 to eliminate SCs of $G_r^{(k)}$ which are not SCs of ${}^{I}G_r^{(k)}$ may be used to eliminate SCs of ${}^{S}G_r^{(k)}$ which are not SCs of ${}^{S+I}G_r^{(k)}$; and the usual argument shows that all SCs of ${}^{S+I}G_r^{(k)}$ are GSs of ${}^{S+I}H$.)

For what values of $r$ and $k$ do we find disordered GSs of ${}^{S+I}H$? Again we just give our conclusions here. Combining the two symmetries

[10] We find that $G_r^{(3)}$ is non-$\mathscr{I}$-disjoint for $r \geqslant 4$ and $G_r^{(k)}$ is non-$\mathscr{I}$-disjoint for any $k \geqslant 4$ and $r \geqslant 2$.

[11] An example is the cycle (0001001101000110111) and its inverse in $G_7^{(2)}$, which has a single-node RBC in ${}^{I}G_7^{(2)}$. We find type (4) SCs for $k = 3$, $r \geqslant 4$ and for ($k = 4$, $r \geqslant 3$), ($k = 5$, $r \geqslant 2$), and ($k = 6$, $r \geqslant 2$).

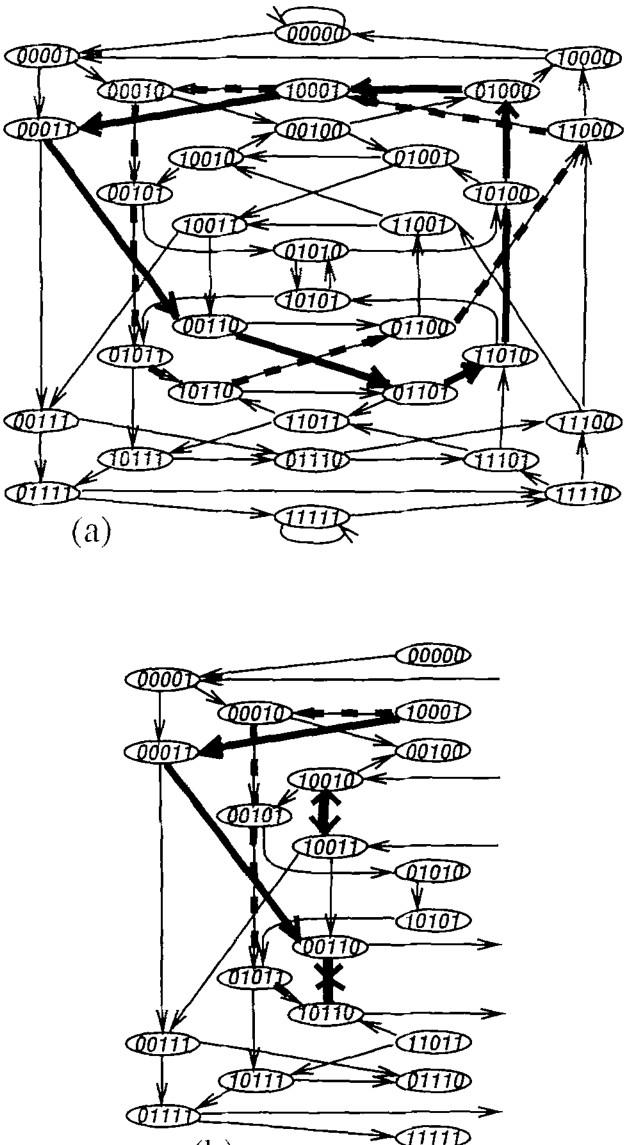



Fig. 9. (a) A D-pair (heavy solid and heavy dashed lines) in $G_5^{(2)}$. Such a pair of cycles gives rise to a degenerate set of configurations, including disordered ones, which are GSs over a finite volume of coupling-parameter space. (b) The appearance of the two SCs of (a) as a type (2) SC in $'G_5^{(2)}$. The two "legs" going from the symmetry line $\mathscr{J}$ to the JA have different "colors" (here represented by solid vs. dashed lines).

eliminates some D-pairs and creates others. With one exception, however, we find that, wherever (in $r$ and $k$—see Table I below) $S$ or $I$ alone gives disordered GSs, the combination $S+I$ also gives disordered GSs. The exception is $G_5^{(2)}$. Here we find (as noted above) that ${}^SG_5^{(2)} \sim G_4^{(2)}$. Since the latter graph has no D-pairs, application of $I$ to ${}^SG_5^{(2)}$ gives a graph ${}^{S+I}G_5^{(2)} \sim {}^IG_4^{(2)}$ which also has no D-pairs. We note finally that $S+I$ does not give disordered GSs where neither $S$ nor $I$ does.

## 6. AN EXAMPLE

We give here a simple example to illustrate the above logic. The simplest case which gives D-pairs (see Table I below) is $k=3$, $r=1$, with $X=S$. We give both $G_1^{(3)}$ and ${}^SG_1^{(3)}$ in Fig. 10; for convenience, we let the three states [formally labeled (0,1,2)] take the values $+1$, 0, and $-1$ (so that $S\sigma = -\sigma$). The dimension of $H$ is $d=6$. The independent correlations are $s_1 = \langle \sigma^2 \rangle$, $s_2 = \langle \sigma_i \sigma_{i+1} \rangle$, $s_3 = \langle \sigma_i^2 \sigma_{i+1}^2 \rangle$, $s_4 = \langle \sigma \rangle$, $s_5 = \langle \sigma_i^2 \sigma_{i+1} \rangle$, and $s_6 = \langle \sigma_i \sigma_{i+1}^2 \rangle$, and the Hamiltonian density is $\mathscr{H} = -\mathbf{J} \cdot \mathbf{s}$.

Since none of the arcs in $G_1^{(3)}$ is constrained by either flow or symmetry, they represent nine distinct densities, giving the nine faces of the polytope $P_1^{(3)}$ which lives in six-dimensional Euclidean space. (We will not give the relationships between the densities and the correlations here; they are readily generalized from those given by Teubner[2] for $k=2$.)

Now we apply $S$ symmetry, and seek disordered GSs. This means $J_4 = J_5 = J_6 = 0$ in $\mathscr{H}$; hence the polyhedron ${}^SP_1^{(3)}$ is the three-dimensional "slice" of $P_1^{(3)}$ given by $s_4 = s_5 = s_6 = 0$. The five arcs of ${}^SG_1^{(3)}$ give rise to five densities: $n_{-0} = n_{+0} \equiv n_{x0}$; $n_{0-} = n_{0+} \equiv n_{0x}$; $n_{+-} = n_{-+} \equiv n_{xx}^{(1)}$; $n_{++} = n_{--} \equiv n_{xx}^{(2)}$; and $n_{00}$. However only four of these are distinct, since the structure of ${}^SG_1^{(3)}$ constrains $n_{0x} = n_{x0}$ by conservation of flow. Hence ${}^SP_1^{(3)}$ is a polyhedron in 3D with four faces: it is a tetrahedron. The vertices of ${}^SP_1^{(3)}$ are the four SCs of ${}^SG_1^{(3)}$: (0), $(+) = (-)$, $(+-)$, and $(0+) = (0-)$. The last is of course the D-pair, sharing the invariant node $\mathscr{N}^* = 0$. Each of these SCs sets three of the four distinct densities to zero, hence shares $d^S = 3$ of the $4(=n_f)$ faces of ${}^SP_1^{(3)}$. In s coordinates, these vertices are, respectively, $(0, 0, 0)$, $(1, 1, 1)$, $(1, -1, 1)$, and $(1/2, 0, 0)$. The SCs $(0-+)$ and $(0+-)$ of $G_1^{(3)}$ are not SCs of ${}^SG_1^{(3)}$; they lie on the edge in ${}^SP_1^{(3)}$ joining $(0+)/(0-)$ to $(+-)$. This is a geometric version of decomposition (recall Fig. 2 and the associated discussion); one can verify graphically that $(0-+)/(0+-)$ decomposes into $(0+)/(0-)$ and $(+-)$.

The (single) D-pair for this case is the lowest energy SC in the subvolume bounded by the planes $J_1/2 + J_2 + J_3 < 0$, $J_1/2 - J_2 + J_3 < 0$, and $J_1 > 0$. For illustration, a representative point in this subvolume is $\mathscr{H} = -\langle \sigma^2 \rangle + \langle \sigma_i^2 \sigma_{i+1}^2 \rangle$ , that is, $J_1 = +1$, $J_2 = 0$, and $J_3 = -1$. For this

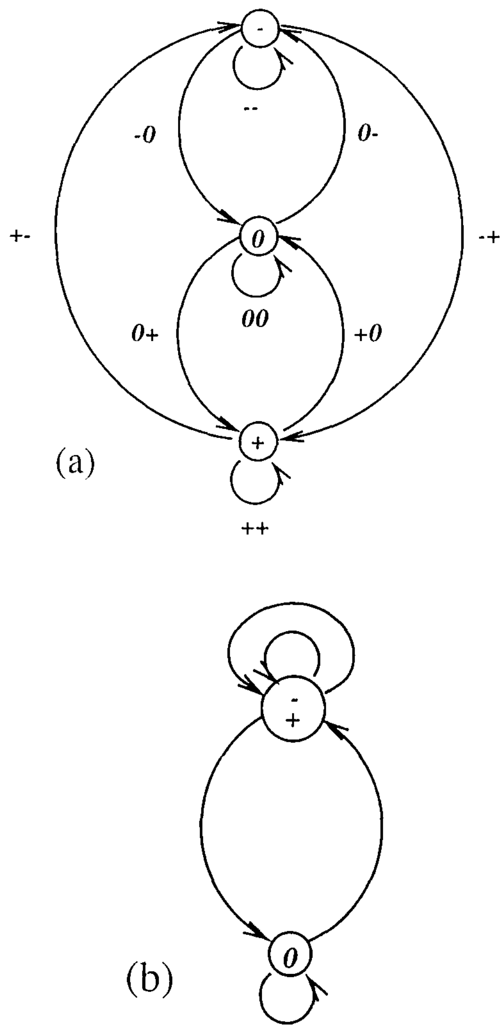



Fig. 10. (a) $G_1^{(3)}$; (b) ${}^SG_1^{(3)}$. In the latter, a pair of energetically distinct arcs connects the node $-/+$ to itself; in general, ${}^SG_r^{(k)}$ has parallel arcs such as these for $k>2$, and so is not a de Bruijn graph.

Hamiltonian (or any other in the subvolume), undefected ground-state configurations are of the form $\ldots 0x0x0x0x0\ldots$, where "$x$" may be, arbitrarily, $+$ or $-$; and the entropy density at $T=0$ is $(\ln 2)/2$.

It would be of some interest to find an analogous neighborhood for the smallest-$r$ Ising ($k=2$) problem with disordered GSs: $r=5$ for $I$

symmetry and $r=6$ for $S+I$ symmetry. This task, however, is considerably more tedious than the above simple example; hence we do not attempt it here.

## 7. ROBUSTNESS OF THE DISORDER

We have defined ground states in such a way as to rule out disorder which may arise at some hypersurface at which there is a multiphase degeneracy, when that hypersurface is of lower dimension than the dimension of $H$. That is, we seek ground states which do not require precise values of couplings; if such GSs are disordered, then the disorder is robust with respect to small variations (of range $r$) in the coupling constants of the Hamiltonian. Imposing this criterion enables us to focus our attention on a finite set of simple cycles of a graph (vertices of a polytope); this set gives us "all" the GSs for a given $k$ and $r$. Such robust GSs are also more likely to be realized experimentally—since in general the couplings in $H$ cannot be controlled to arbitrary precision in experiment.

There are other criteria for robustness of a GS—that is, other perturbations which can be considered. In this section we will briefly consider two further types of perturbation: interactions of range $r' > r$ and temperature-driven fluctuations. For each such perturbation we wish to examine the stability of a disordered GS of ${}^{X}H_r^{(k)}$.

### 7.1. Longer Ranged Interactions

We first consider interactions of range $r' > r$. Such interactions were considered to be strictly zero in our preceding arguments; we now consider them to be very small, but nonzero. Specifically, let us allow all ($X$-invariant) couplings of range $r+1$ to be small but nonzero. By our above logic, we now face a new problem, for which we must study the SCs of ${}^{X}G_{r+1}^{(k)}$, or equivalently, the vertices of ${}^{X}P_{r+1}^{(k)}$. A spin configuration which is a GS configuration for ${}^{X}H_r^{(k)}$ is then stable under the perturbation if and only if the same configuration is a GS configuration for ${}^{X}H_{r+1}^{(k)}$.

Given that ${}^{X}P_{r+1}^{(k)}$ is convex and that the projection $(r+1) \Rightarrow r$ is a linear operator, one can show that any vertex $v$ of ${}^{X}P_r^{(k)}$ is the projection of one or more vertices of ${}^{X}P_{r+1}^{(k)}$. Now assume that $X=0$, i.e., we consider the generic case. In this case we have that the arc densities of each vertex of $P_r^{(k)}$ are uniquely determined (they are all $1/p$ or 0, where $p$ is the period of the SC). This in turn determines the node densities, and hence the arc densities, at range $r+1$. Thus we see that each vertex of $P_r^{(k)}$ has a single preimage in ${}^{X}P_{r+1}^{(k)}$ under the projection. In other words, any SC of $G_r^{(k)}$ is also an SC of $G_{r+1}^{(k)}$ (as may be proved by simpler means).

For the symmetry-constrained case $X \neq 0$, vertices of ${}^{X}P_r^{(k)}$ can represent multiple sets of arc densities (although they correspond to a unique set of *distinct* arc densities). This happens when the corresponding SC of ${}^{X}G_r^{(k)}$ maps to a pair of SCs in $G_r^{(k)}$. Hence we consider a vertex $v$ of ${}^{X}P_r^{(k)}$ corresponding to a degenerate pair of SCs of $G_r^{(k)}$. Suppose that $v$ has multiple preimages $\{v'\}$ in ${}^{X}P_{r+1}^{(k)}$. Then the correlations represented by $v$ (with those of range $r+1$ set of zero) lie, not on a vertex of ${}^{X}P_{r+1}^{(k)}$, but rather at the locus of degeneracy of the vertices $\{v'\}$. Therefore, $v$ has multiple preimages only if interactions of range $r+1$ reduce the degeneracy associated with $v$.

Suppose the pair of SCs associated with $v$ shares no nodes in $G_r^{(k)}$ (so that we have SSB). Then the degeneracy of the spin configurations at $v$ is subextensive (as discussed in Section 2), being generated entirely (that is, after counting the two periodic configurations) by defects of vanishing density. This degeneracy will not be reduced by any symmetric perturbation of range $r' > r$. The vertex corresponding to such a pair will then have a single preimage in ${}^{X}P_{r+1}^{(k)}$. One can easily show that this latter vertex [in ${}^{X}P_{r+1}^{(k)}$] corresponds to the same pair of SCs; hence all the GS spin configurations are stable to the perturbation.

We now consider a D-pair in $G_r^{(k)}$ corresponding to a single vertex $v$ in ${}^{X}P_r^{(k)}$. If a D-pair shares $n$ nodes in $G_r^{(k)}$, it will share $n-1$ nodes in $G_{r+1}^{(k)}$ [and $\max(n-m, 0)$ nodes in $G_{r+m}^{(k)}$]. When the number of shared nodes in $G_{r+1}^{(k)}$ is nonzero the D-pair remains a D-pair; the degeneracy is unchanged; and so $v$ is the projection of a single vertex of ${}^{X}P_{r+1}^{(k)}$. However, when the number of shared nodes reaches zero, the domain-wall energy is no longer exactly zero. For positive domain-wall energy, we have SSB: $\mathscr{H}$ is minimized at a vertex $v_+$ of ${}^{X}P_{r+1}^{(k)}$, representing a degenerate pair of undefected periodic ground-state configurations (represented by, say, $cyc$ and $\overline{cyc}$), plus a set of defected configurations of vanishing defect density and entropy density. When the domain-wall energy is negative, the undefected GS is a regular crystal of domain walls, which appears in ${}^{X}P_{r+1}^{(k)}$ as a vertex $v_-$ representing a single SC $(cyc \cdot \overline{cyc})$ of period $2p$. Both vertices $v_+$ and $v_-$ map to the single (D-pair) vertex $v$ in ${}^{X}P_r^{(k)}$—since, up to interactions of range $r$, the domain-wall energy is exactly zero, and the configurations represented by $v_+$ and $v_-$ are degenerate. The ground-state configurations corresponding to $v_+$ and $v_-$ have, however, a vanishing entropy density, whereas their single image $v$ in $P_r^{(k)}$ represents a set of configurations with finite entropy density. Therefore, almost all of these latter configurations are unstable under the perturbation.

Hence we find that a D-pair of ${}^{X}H_r^{(k)}$ is stable *as a D-pair* to perturbations of range $r+1$ if and only if it includes more than one shared node. Our example in Section 6 illustrates the possible instability of a D-pair: the vertex of ${}^{S}P_1^{(3)}$, corresponding to the D-pair $(0+)/(0-)$ of $G_1^{(3)}$, appears as two vertices of ${}^{S}P_2^{(3)}$, namely, the SSB pair $(0+)/(0-)$ and the domain-wall crystal $(0+0-)$. This example is representative of all the D-pairs arising from $S$ symmetry; since they always share a single node at range $r$, they are always unstable to small interactions of range $r+1$.

Problems with $I$ symmetry allow for D-pairs with more than one shared node. (An example is the cycle 000001001101110001011 and its $I$-partner, which share two nodes in $G_6^{(2)}$.) Such D-pairs are stable—as D-pairs—to small perturbations of range $r+1$, but of course are not stable—as D-pairs—to perturbations of arbitrary range.

Summarizing the above, we find that some spin configurations represented by a D-pair at range $r$ always occur as GS configurations for any $r' > r$. However, the *extensive entropy* of the D-pair is *not* stable to arbitrary $r$; it is reduced to a vanishing entropy density by interactions of sufficient range to render the domain-wall energy nonzero. Specifically, D-pairs sharing $n$ nodes at range $r$ cease to be D-pairs at range $r+n$; at this range and beyond, the degeneracy of the remaining configurations is subextensive, and the GSs are periodic.

## 7.2. Temperature

We now consider the case that the temperature $T$ is small but not zero. This case merits consideration since it is known that a small temperature can act analogously to the small couplings at $r+1$ above. That is, a small temperature can select out a subset of the set of degenerate ground states such that there is a nonzero order parameter over the selected subset of states. This order parameter hence only vanishes at $T$ exactly zero, and not in the limit $T \to 0^+$. This kind of behavior has been called "order from disorder."[12]

The (temperature-driven) selection comes about because those excitations which dominate at low $T$ (due to low energy and/or high density of states) have significant overlap with ("select") only a subset of the degenerate ground states. The order parameter then appears because the selected subset may be distinguished from the nonselected subset; crudely speaking, only an ordered subset of the degenerate ground states gives rise to the dominant low-energy excitations.

The D-pairs we have identified represent highly degenerate ground states. It is then natural to ask, might temperature select out, from this disordered set of ground states, an ordered subset? In other words, is the disorder unstable against a small perturbation in $T$?

Our answer is no: the disorder is stable to small $T$. The reasoning is simple. Every undefected GS configuration in the set represented by a

D-pair consists of a periodic array of the shared nodes, with (arbitrarily) a structure $s$ or its inverse $\bar{s}$ between the shared nodes. Since the spin states are discrete, the low-energy excitations are localized: they are the defects discussed in Section 2. A defect may be viewed as replacing *one* intervening $s$ or $\bar{s}$ with some other spin sequence, not necessarily of the same length. Clearly, such excitations leave the remainder of the chain as free to sample $s$ or $\bar{s}$ as it was at $T=0$. Hence there is no order at low $T$ that is not present at $T=0$.

An important element of our reasoning is the fact that our problem is one-dimensional. Order from disorder was shown to occur for discrete spins in two dimensions by Villain *et al.*[12]

## 8. DISCUSSION AND SUMMARY

We summarize our findings in Table I. Each $X$ entry means that there are D-pairs in $G_r^{(k)}$ arising from the the appropriate types of SCs in ${}^XG_r^{(k)}$. This in turn means that for the given $r$ and $k$, the corresponding $k$-state problem has degenerate, disordered GSs, arising from $X$ symmetry, over a finite region of coupling-parameter space, without any fine tuning (beyond that coming from the symmetry).

Table I is striking in the near-ubiquity of its entries, which stands in strong contrast to the simple result of Radin and Schulman.[1] Given that $I$ symmetry is ubiquitous as well, we might expect disordered GSs for a number of interesting problems. However, both the present results and those of RS remain somewhat academic in the absence of a convincing

**Table I. One-Dimensional $k$-State Models Which Allow for Disordered Ground States, with No Fine Tuning of Coupling Parameters Other Than That Demanded by Symmetry[a]**

| | $r=1$ | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $k=2$ | | | | | $I$ | $I$ | $\cdots$ |
| 3 | $S$ | $S, I$ | $S, I$ | $S, I$ | $S, I$ | $S, I$ | $\cdots$ |
| 4 | | $I$ | $I$ | $I$ | $I$ | $I$ | $\cdots$ |
| 5 | $S$ | $S, I$ | $S, I$ | $S, I$ | $S, I$ | $S, I$ | $\cdots$ |
| 6 | | $I$ | $I$ | $I$ | $I$ | $I$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

[a] Rows are indexed by the number of states $k$, and columns by the range $r$ (in lattice constants) of the interactions. An entry is made whenever spin inversion ($S$) or spatial inversion ($I$) symmetry gives rise to one or more pairs of degenerate, periodic ground states, with zero surface tension between the two members of the pair. It is this combination (degeneracy plus zero surface tension) which allows for disordered ground states.

physical application. It may or may not be the case that the regions of parameter space giving disordered GSs are in general too "weird" (i.e., unphysical) to be visited by physical problems. Only further work can answer this question. The answer is, however, of considerable interest since it has relevance to the third law of thermodynamics:[13] as noted above, a D-pair represents a ground state with *extensive* entropy (finite entropy density) at $T=0$.

We note finally that there is at least one physically motivated problem, namely the problem of stacking polytypes in crystals (see, e.g., ref. 14), which should be well modeled by an effective, one-dimensional Ising Hamiltonian with medium-[15] or long-range[16] effective interactions between the stacking units (layers). Materials showing polytypism do show very long period and disordered structures, even down to low temperatures. It is likely that the disordered structures are metastable configurations, trapped at low temperature by a "rugged" energy surface. However, given our present results (and the $S+I$ symmetry of the problem), we believe that the possibility that some of the disordered structures are ground states cannot be ruled out *a priori*; hence such a possibility deserves further study.

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